Effective time variation of G in a model universe with variable space dimension

Reza Mansouri[a,b], Forough Nasseri[a,b], Mohammad Khorrami[a,c]

[a] *Institute for Studies in Theoretical Physics and Mathematics, P.O.Box 19395–5531, Tehran, Iran*
[b] *Department of Physics, Sharif University of Technology, P.O.Box 11365–9161, Tehran, Iran*
[c] *Institute for Advanced Studies in Basic Science, P. O. Box, 45195–159, Gava Zang, Zanjan, Iran*

Abstract

Time variation of Newtonian gravitational constant, G, is studied in model universes with variable space dimension proposed recently. Using the Lagrangian formulation of these models, we find the effective gravitational constant as a function of time. To compare it with observational data, a test theory for the time variation of G is formulated. We have assumed a power law behavior of the time variation of G where the exponent β is itself time dependent. Within this test theory we are able to restrict the free parameter of the theories under consideration and give upper bounds for the space dimension at the Planck era. The time variation of G at earlier times, such as the time of nucleosynthesis is also predicted which express the needs to look for related observational data.

1. Introduction

The present value of Newtonian gravitational constant, G, is known with the least accuracy [1,2]. There are three problems connected with G: absolute value measurements and possible variations with time [3,4] and space [5,6]. There is a promising new space experiment SEE–Satellite Energy Exchange– [7] which addresses to all these problems and may be more effective in solving them than other laboratory or space experiments. The SEE project may improve our knowledge of G, limits on temporal and space variation by $2-3$ orders of magnitude [8].

Current ideas suggest that the value of G might be related to other fundamental constants of physics [9]. It has been recently suggested that gravity could be even more drastically modified below some distance, r_0. In principle, Cavendish–type experiments performed for separation smaller than r_0 might see a change of the $1/r^2$ law: the exponent 2 being replaced by an exponent larger than or equal to 4. In these models, r_0 is already constrained to be smaller than $\sim 1\mu m$ [10]. G–measurements and Cavendish–type experiments have now reached a new significance as possible windows on the physics of unification between gravity and the other interactions.

It must be noted that the variability of G is explicitly or effectively model dependent. In several modified quantum theories of gravity, scaling of G with the distance is standard [11–14]. However, the modification of G is small and Newtonian gravity holds in the weak energy limit. Confirmations of this scheme are coming from satellites' measurements of long acceleration [15].

Interest in the problem of time variation of G has increased greatly during the last decade because of new developments in Kaluza–Klein and superstring theories of the unification of all physical interactions. Observational bounds on the time evolution of extra spatial dimensions in Kaluza–Klein and superstring theories can be obtained from limits on possible variation of G and other constants [16–19]. The recent version of the dilaton evolution proposed by Damour and Polyakov [20] in the context of string theory has provided an expression connecting the time variation of G and that of fine structure constant, α. Barrow [21] has discussed the variation of G in Newtonian and relativistic scalar–tensor gravity theories.

In this paper, we will discuss on the effective time variation of G in the models proposed in [22,23,25]. In these models, the space dimension varies with the expansion of the universe. The model proposed in [22,23] seems to be singularity free, having two turning points for the space dimension. This model has been criticized in [24]. The way of generalizing the standard cosmological model to arbitrary space dimension used in [22,23] is questioned, and another way of writing the field equations is proposed. It has been pointed out that the model given in [22,23] has no upper bound for the dimension of space.

Later on, we studied critically the previous works in [22–24], and derived new Lagrangians and field equations. We also discussed the model universe with variable space dimension from the view point of quantum cosmology and obtained a general wave function for this model. In the limit of constant space dimension, our wave function approaches the tunneling Vilenkin wave function or the modified Linde wave function [25].

Here we are interested to study the time variation of G within these cosmological models with variable space di-

mension. There is a hope that observational data for time variation of G may distinguish between these theories or show their viability.

Every comparison of a theory with experiment or observation needs a socalled test theory (for examples in special relativity see [26]). Such a formulation for G variation is given in [27], which is however not general enough to include models we are interested in. Therefore a new generalized test theory for time variation of G is formulated in section 2. A short review of the cosmological models with variable space dimension is given in section 3. Based on the test theory of section 2, we are able to restrict the free parameter of the theories under consideration to be consistent with observation. This gives us upper bounds for the dimension of space at the Planck era. Our generalized test theory allows to predict the variation of G at even earliest times. Therefore, the value of G at the time of nucleosynthesis can be compared to that of the present ime.

2. Formulation of a one parameter test theory for the time variation of G

It is always very fruitful to have a theory for any tests of physical theories or constants (see for example [26] for an extensive use of test theories in special relativity). Test theories are not only helpful to interpret the experimental or observational data but they also direct us to new and sometimes crucial tests. Therefore one should care about their formulations. Here we will elaborate on an existing formulation for the time variation of G [27] and generalize it to more physical cases. Let us take

$$G(t) = G_0(\frac{t}{t_0})^{\beta}, \tag{1}$$

where $t_0 \simeq 10^{17}$sec is the present time and G_0 is the present value of G. In this case, the average rate of the cosmological time variation of G is

$$\frac{\dot{G}}{G} = \frac{\beta}{t}, \tag{2}$$

so that today

$$\left(\frac{\dot{G}}{G}\right)_0 = \frac{\beta}{t_0}. \tag{3}$$

Data from big bang nucleosynthesis yields

$$|\beta| \lesssim 0.01, \tag{4}$$

or $(\dot{G}/G) \lesssim 10^{-12}\mathrm{yr}^{-1}$ [27]. We now generalize the relation (1) to the cases where β is not constant but a function of time:

$$G = G_0(\frac{t}{t_0})^{\beta(t)}. \tag{5}$$

Time derivative of this equation yields

$$\frac{\dot{G}}{G} = \frac{\beta(t)}{t} + \dot{\beta}(t)\ln(\frac{t}{t_0}). \tag{6}$$

Now, if $\beta(t)$ and its time derivative, $\dot{\beta}(t)$, satisfy the following condition

$$|\frac{\dot{\beta}(t)}{\beta(t)}| \ll |\frac{1}{t\ln\frac{t}{t_0}}|, \tag{7}$$

Eq.(6) can be written as

$$\frac{\dot{G}}{G} \simeq \frac{\beta(t)}{t}, \tag{8}$$

which looks like (3) with time dependent β. The condition (7) may not always be valid. Therefore it must be checked for each case. This test theory shows for example how time variation of G could be different in different eras. We are then led to find observation giving data for time variation of G in early universe or later times. We will see in the next section that models with variable space dimensions can be handled within this new test theory.

3. Review of the model universe with variable space dimension

The model introduced in [22,23] is based on a flat Friedmann universe with dynamical space dimension. Here we do not restrict ourselves to a special topology and introduce $k = -1, 0, +1$ for the open, flat, or closed model, respectively. The Lagrangian of our model universe is

$$\mathcal{L} := -\frac{D(D-1)}{2\kappa N}[(\frac{\dot{a}}{a})^2 - \frac{N^2k}{a^2}](\frac{a}{a_0})^D + \frac{1}{2}(-\hat{\rho}N^2 + \hat{p}Da^2), \tag{9}$$

where

$$\hat{\rho} := \frac{rho}{N}(\frac{a}{a_0})^D, \hat{p} := pa^{-2}N(\frac{a}{a_0})^D.$$

There is a constraint in this model which can be written as

$$(\frac{a}{\delta})^D = (\frac{a_0}{\delta})^{D_0} = e^C, \tag{10}$$

or

$$\frac{1}{D} = \frac{1}{C}\ln(\frac{a}{a_0}) + \frac{1}{D_0}. \tag{11}$$

Here is a the scale factor of the Friedmann universe, N the lapse function, D the variable space dimension, ρ the energy density, p the pressure, δ the characteristic minimum length of the model, C a constant of the model,

and $\kappa = 8\pi G$. The zero subscript in any quantity, e.g. in a_0 and D_0, denotes its present value. Note that in Eqs.(9, 10), the space dimension is a function of cosmic time, t. Time derivative of Eq.(10) leads to

$$\dot{D} = -\frac{D^2\dot{a}}{Ca}. \tag{12}$$

It is easily seen that the case of constant space dimension corresponds to the limit of $C \to +\infty$. Varying the action (9) with respect to N and a, and taking $\hat{\rho}$ and $\hat{p}$ as constant, we arrive at the following equations of motion in the gauge $\dot{N} = 0$

$$\frac{1}{N^2}(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{2\kappa\rho}{D(D-1)}, \tag{13}$$

and

$$\frac{\ddot{a}}{a} + [\frac{D^2}{2D_0} - 1 - \frac{D(2D-1)}{2C(D-1)}]\{(\frac{\dot{a}}{a})^2 + \frac{N^2k}{a^2}\} + N^2\kappa p(\frac{1-\frac{D}{2C}}{D-1}) = 0. \tag{14}$$

Using these equations of motion, one can easily obtain the continuity equation

$$\frac{d}{dt}(\rho(\frac{a}{a_0})^D) + p\frac{d}{dt}(\frac{a}{a_0})^D = 0. \tag{15}$$

In the limit of constant space dimension, or $C \to +\infty$, Eqs. (9, 13, 14, 15) approach to the corresponding equations for constant space dimension $D = D_0$:

$$\mathcal{L}^0 := \{-\frac{D_0(D_0-1)}{2\kappa_0 N}[(\frac{\dot{a}}{a})^2 - \frac{N^2k}{a^2}] + \frac{N}{2}(-\rho + Dp)\}(\frac{a}{a_0})^{D_0}, \tag{16}$$

$$\frac{1}{N^2}(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{2\kappa_0\rho}{D_0(D_0-1)}, \tag{17}$$

$$\frac{\ddot{a}}{a} + (\frac{D_0-2}{2})\{(\frac{\dot{a}}{a})^2 + \frac{N^2k}{a^2}\} + \frac{N^2\kappa_0 p}{D_0-1} = 0, \tag{18}$$

$$\frac{d}{dt}(\rho a^{D_0}) + p\frac{d}{dt}(a^{D_0}) = 0. \tag{19}$$

We have introduced κ_0 for the value of the gravitational coupling constant in the case of the constant space dimension $D = D_0$. In Ref. [25], we have mentioned some of the shortcomings of the original model proposed in [22,23], regarding the field equations and their results. There, we have mentioned that the Lagrangian is not unique. Using Hawking–Ellis action of a prefect fluid, we have obtained the following Lagrangians for the model universe with variable space dimension:

$$\mathcal{L}_I := -\frac{V_D}{2\kappa}(\frac{a}{a_0})^D\frac{D(D-1)}{N}\{(\frac{\dot{a}}{a})^2 - \frac{N^2k}{a^2}\} - \rho N V_D(\frac{a}{a_0})^D, \tag{20}$$

and

$$\mathcal{L}_{II} := -\frac{V_D}{2\kappa}(\frac{a}{a_0})^D\{\frac{2\dot{D}\dot{a}}{aN} + \frac{2D\dot{a}\dot{D}}{aN}\ln\frac{a}{a_0} + \frac{D(D-1)}{N} \times\{(\frac{\dot{a}}{a})^2 - \frac{N^2k}{a^2}\} + \frac{2D\dot{D}\dot{a}}{Na}\frac{d\ln V_D}{dD}\} - \rho V_D N(\frac{a}{a_0})^D. \tag{21}$$

Here V_D is the volume of the spacelike sections:

$$V_D = \begin{cases} \frac{2\pi^{(\frac{D+1}{2})}}{\Gamma(\frac{D+1}{2})}, & \text{if } k = +1, \\ \frac{\pi^{(\frac{D}{2})}}{\Gamma(\frac{D}{2}+1)}\chi_c{}^D, & \text{if } k = 0, \\ \frac{2\pi^{(\frac{D}{2})}}{\Gamma(\frac{D}{2})}f(\chi_c), & \text{if } k = -1, \end{cases} \tag{22}$$

where χ_c is a cut–off and $f(\chi_c)$ is a function thereof (see Ref. [25]). In the limit of constant space dimension, $\mathcal{L}_I$ and $\mathcal{L}_{II}$ approach to

$$\mathcal{L}^0_{I,II} := -\frac{V_{D_0}}{2\kappa_0 N}(\frac{a}{a_0})^{D_0}D_0(D_0-1)[(\frac{\dot{a}}{a})^2 - \frac{N^2k}{a^2}] - \rho N V_{D_0}(\frac{a}{a_0})^{D_0}. \tag{23}$$

A complete discussion of the field equations corresponding to $\mathcal{L}_I$ and $\mathcal{L}_{II}$, the dynamics of the model, and the wave function of the universe have been given in Ref. [25].

We define D_{Pl} as the space dimension of the universe when the scale factor is equal to the Planck length, l_{Pl}. Taking the scale of universe at $D_0 = 3$, to be the present value of the Hubble radius, H_0^{-1}, and the space dimension in the Planck length to be 4, 10, or 25, coming from Kaluza–Klein and superstring theories, we can obtain from Eqs.(10,11), the corresponding value of C and δ, see Table I.

TABLE I. Values of C and δ for some interesting values of D_{Pl} assuming $D_0 = 3$.

C	D_{pl}	δ(cm)
∞	3	0
1678.8	4	8.6×10^{-216}
599.57	10	1.5×10^{-59}
476.93	25	8.4×10^{-42}
419.70	$+\infty$	l_{Pl}

4. Time variation of G in model universe with variable space dimension

Take first the model based on $\mathcal{L}$. The gravitational coupling constant κ have to be related to G. Although this relation seems not to be trivial [28], we will assume for simplicity the familiar relation $\kappa = 8\pi G$. Now, comparing the coefficients of $(\dot{a}/a)^2$ in $\mathcal{L}^0$ and $\mathcal{L}$ we obtain

$$\frac{G}{D(D-1)} = \frac{G_0}{D_0(D_0-1)}. \tag{24}$$

This may also be written as

$$G = G_0 f(D), \tag{25}$$

where

$$f(D) := \frac{D(D-1)}{D_0(D_0-1)}. \tag{26}$$

Time derivative of Eq.(25) yields

$$\frac{\dot{G}}{G} = \dot{D}\frac{f'(D)}{f(D)} = -\frac{D^2 \dot{a} f'(D)}{Caf(D)}, \tag{27}$$

where prime denotes the derivative with respect to D. To express the time evolution of G given by Eq.(27) in terms of t, we have to know a and D as functions of the time. This requires the solution of the equation of motion (13, 14) which does not seem to have any analytic solution. As an approximation, we take the solution of the standard big bang cosmology in the radiation dominated (RD) and matter dominated (MD) universe for the constant D space dimension:

$$a = a_0(\frac{t}{t_0})^{1/2}, \Rightarrow a = a_0(\frac{t}{t_0})^{2/(D+1)}, \text{ RD universe}, \tag{28}$$

$$a = a_0(\frac{t}{t_0})^{2/3}, \Rightarrow a = a_0(\frac{t}{t_0})^{2/D}, \text{ MD universe}. \tag{29}$$

Assuming now D to be variable, and using (12), the time derivative of Eqs.(28) and (29) yield, respectively:

$$\frac{\dot{a}}{a} = \frac{2}{t[1+\frac{D^2}{D_0}]}, \quad \text{RD universe}, \tag{30}$$

$$\frac{\dot{a}}{a} = \frac{2D_0}{D^2 t}, \quad \text{MD universe}. \tag{31}$$

Inserting (30, 31) in (27), we obtain the time evolution of G for the radiation–dominated universe

$$\frac{\dot{G}}{G} \simeq \frac{\beta_{RD}}{t}, \tag{32}$$

and for the matter–dominated universe

$$\frac{\dot{G}}{G} \simeq \frac{\beta_{MD}}{t}, \tag{33}$$

where

$$\beta_{RD} = -\frac{2D^2 f'(D)}{C[1+\frac{D^2}{D_0}]f(D)}, \tag{34}$$

$$\beta_{MD} = -\frac{2D_0 f'(D)}{Cf(D)}. \tag{35}$$

We may take β_{RD} or β_{MD} given by (34) or (35) for $\beta(t)$ in Eqs.(6,7,8). But first notice that for the present time t_0, the second term in the RHS of Eq.(6) vanishes and Eq.(8) is exactly true. The present value of $\beta(t)$, say β_0, is obtained by inserting $D = D_0 = 3$ in Eq.(35). Note that β_0 depends on the C–parameter and consequently on D_{Pl}. Table II shows some values of β_0.

The earliest time for which we have observational data is the time of nucleosynthesis, $t_{ns} \sim 1$sec. Using the standard big bang cosmology, the scale factor of the universe at the nucleosynthesis time is nearly $a_{ns} \sim 3 \times 10^{18}$cm. The corresponding value of the space dimension, D_{ns}, can be obtained by inserting $a = a_{ns}$, $a_0 = H_0^{-1} \simeq 10^{28}$cm and $D_0 = 3$ in Eq.(11). For each value of C or D_{Pl}, the corresponding value of D_{ns} is given in Table II. The value of β at the time of nucleosynthesis, β_{ns}, is now obtained from (34) using the appropriate D_{ns} value. It can now be easily seen that the condition (7) is still valid for $t = t_{ns}$. Therefore, Eq.(8) can be used for the time variation of G at t_{ns}. Numerical values for β_{ns} and G_{ns}/G_0 are obtained for some values of D_{Pl} in Table II. We are not aware of any observational data relating to this quantity, but given observational data one can look for the validity of the underlined theory.

Let us now compare the numerical value of β_0 for the model $\mathcal{L}$ with observational data, $|\beta| \lesssim 0.01$, see Eq.(4). From Table II, we see that for $D_{Pl} > 25$, the corresponding absolute values of β_0 are bigger than 0.01. Hence, models defined by $\mathcal{L}$ having $D_{Pl} > 25$ are ruled out.

Let us now consider the Lagrangians $\mathcal{L}_I$ and $\mathcal{L}_{II}$. In this case, the comparison of the coefficients of $(\dot{a}/a)^2$ in $\mathcal{L}_I$ and $\mathcal{L}_{II}$ with $\mathcal{L}^0_{I,II}$ give us the relation between G_0 and G:

$$\frac{G}{V_D D(D-1)} = \frac{G_0}{V_{D_0} D_0(D_0-1)}, \tag{36}$$

which can be written as

$$G = F(D) G_0, \tag{37}$$

where

$$F(D) := \frac{V_D D(D-1)}{V_{D_0} D_0(D_0-1)}. \tag{38}$$

For simplicity, let us now assume a closed Fridmann universe with $k = +1$. From Eq.(38) we then obtain

$$F(D) = \frac{\pi^{D/2} D(D-1)\Gamma(\frac{D_0+1}{2})}{\pi^{D_0/2} D_0(D_0-1)\Gamma(\frac{D+1}{2})}. \tag{39}$$

Similar to our treatment for $\mathcal{L}$, it can be shown that for the Lagrangians $\mathcal{L}_I$ and $\mathcal{L}_{II}$, the time evolution of G is given by Eq.(5), where β is defined according to (34,35) except that $f(D)$ is replaced by $F(D)$. It is also seen that the condition (7) is still valid for the time of nucleosynthesis. Some interesting values for β_0, and β_{ns}, and G_{ns}/G_0 are given in Table III.

Comparing the values of β_0 from Table III with observational data, given by Eq.(4), models with $D_{Pl} = 10, 25,$

and so are ruled out.

TABLE II. Values of C, D_{Pl}, D_{ns}, β_0, β_{ns}, G_{ns}/G_0, based on $\mathcal{L}$ and $\mathcal{L}^0$. The limit of $C \to +\infty$ corresponds to the constant space dimension.

C	D_{Pl}	D_{ns}	β_0	β_{ns}	G_{ns}/G_0
$+\infty$	3	3	0	0	1
1678.8	4	3.12	-0.003	-0.0021	1.09
599.57	10	3.37	-0.008	-0.0057	1.26
476.93	25	3.48	-0.010	-0.0070	1.32
419.70	$+\infty$	3.56	-0.012	-0.0078	1.33

TABLE III. Values of C, D_{Pl}, β_0, β_{ns}, G_{ns}/G_0 based on $\mathcal{L}_I$, $\mathcal{L}_{II}$, and $\mathcal{L}^0_{I,II}$.

C	D_{Pl}	β_0	β_{ns}	G_{ns}/G_0
$+\infty$	3	0	0	1
1678.8	4	-0.004	-0.003	1.13
599.57	10	-0.012	-0.008	1.38
476.93	25	-0.015	-0.010	1.49
419.70	$+\infty$	-0.017	-0.011	1.50

5. Conclusions

The time evolution of G in model universes with variable space dimension has been studied. Assuming a power law behavior for the time variation of G, it turns out that the exponent β has to be time dependent to generalize enough to include models with variable space dimension. This has led us to a generalized test theory for the time variation of G. Within this new test theory of G variation, we have shown that theories based on $\mathcal{L}$ are observationally viable for $D_{Pl} \leq 25$. Those theories based on $\mathcal{L}_I$ and $\mathcal{L}_{II}$ are observationally ruled out for $D_{Pl} \geq 10$.
Within this generalized test theory, it is possible to predict the time variation of G at the era of nucleosynthesis for which it is possible to have observational data. Therefore, we suggest to look for differences in the value of G at the present time and the time of nucleosynthesis, and compare it to the prediction of theories within this test theory.

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